

Weak Poincaré Inequality for Convolution Probability Measures

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Abstract. Weak Poincaré inequalities are established for convolution measures by using Lyapunov conditions, and new Lyapunov function is constructed. The stability for weak Poincaré inequalities under convolution with compactly supported probability measures are discussed. Detailed examples are included to show the power of the main result.

Keywords: Weak Poincaré inequality, Lyapunov condition, convolution.

1 Introduction and main result

It is well known that functional inequalities of Dirichlet forms are powerful tools in characterizing the properties of Markov semigroups and their generators, see [11] for instance. To deal with generators with less regular or less explicit coefficients, an efficient way is to regard the measures as perturbations from better ones, which satisfy the underlying functional inequalities. So it is important to investigate perturbations of functional inequalities.

The convolution probability measure, in the sense of independent sum of random variables, can be regarded as a kind of perturbation; see e.g. [5, 13] and reference therein. In [12], explicit sufficient conditions have been presented such that the convolution probability measures satisfy the log-Sobolev/ Poincaré/ super Poincaré inequalities. The present article is thus a continuation of [12] for the study of weak Poincaré inequalities for the convolution probability measures.

We say that a probability measure μ satisfies the *weak Poincaré inequality* if for all $f \in C_b^1(\mathbb{R}^d)$ with $\mu(f) = 0$, there exists a nonnegative and decreasing function α on $(0, r_0)$, $r_0 < \infty$ such that

$$\|f\|^2 \leq \alpha(r)\mu(|\nabla f|^2) + r\text{Osc}^2(f), \quad 0 < r < r_0, \quad (1.1)$$

where $\text{Osc}(f) = \sup_{x,y \in M} |f(x) - f(y)|$ (see [10] for a general form). Indeed, weak Poincaré inequalities have many applications. In the symmetric case, they describe the decay of the semigroup P_t associated to Dirichlet form (see [1, 10]). Namely for all bounded centered function f , there exists $\psi(t)$ tending to zero at infinity such that $\|P_t f\| \leq \psi(t)\|f\|_\infty$. Another application is the concentration of measure phenomenon for sub-exponential laws (see [2]). From these, we find that it suffices for us to measure the rate of $\alpha(r)$ when $r \rightarrow 0$. So, in (1.1), we only have to consider $r < r_0$ for some positive constant $r_0 > 0$ directly.

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Let μ and ν be two probability measures on \mathbb{R}^d . The perturbation of μ by a probability measure ν on \mathbb{R}^d is given by their convolution

$$(\nu * \mu)(A) := \int_{\mathbb{R}^d} 1_A(x+y) \mu(dx) \nu(dy),$$

where $A \in \mathcal{B}(\mathbb{R}^d)$. Throughout this paper, let $\mu(dx) = e^{-V(x)} dx$ be a probability measure on \mathbb{R}^d such that $V \in C^1(\mathbb{R}^d)$. For the probability measure ν on \mathbb{R}^d , we define

$$p_\nu(x) = \int e^{-V(x-z)} \nu(dz), \quad V_\nu(x) = -\log p_\nu(x), \quad x \in \mathbb{R}^d.$$

Then

$$(\mu * \nu)(dx) = p_\nu(x) dx = e^{-V_\nu(x)} dx. \quad (1.2)$$

From (1.2), we know that the density $\mu * \nu$ with respect to dx is strictly positive, then by [10], weak Poincaré inequalities for $\mu * \nu$ hold automatically. Therefore, the aim of this article is to provide the estimation of the function α in the weak Poincaré inequality for the measure $\mu * \nu$.

It is well known that Lyapunov conditions furnish some results on the long time behavior of the laws of Markov processes (see e.g. [7, 8, 9] and references therein). The relationship between Lyapunov conditions and functional inequalities of Poincaré type (ordinary or weak or super Poincaré) has already been studied in [1, 4]. Here, we first use a Lyapunov condition to obtain a weak Poincaré inequality for $\mu * \nu$, and then present explicit construction of Lyapunov functions to obtain a general result, which is presented as follows. To save words, we introduce some notations here. For each positive function ϕ on \mathbb{R}^d , let

$$\beta_\phi(r) = \mu \left(|x| \geq \frac{1}{2} \varphi_\phi(r) \right) + \nu \left(|x| \geq \frac{1}{2} \varphi_\phi(r) \right), \quad (1.3)$$

where $\varphi_\phi(r) = \sup\{s > 0 : \inf_{|x| \leq s} \phi \geq 1/r\}$. Moreover, let

$$\nu_x(dz) = \frac{1}{p_\nu(x)} e^{-V(x-z)} \nu(dz), \quad x \in \mathbb{R}^d.$$

Theorem 1.1. *Let $\mu(dx) = e^{-V(x)} dx$ be a probability measure on \mathbb{R}^d and ν be a probability measure on \mathbb{R}^d .*

(a) *If $V \in C^1(\mathbb{R}^d)$ satisfies that there exists $R_0 > 0$ such that*

$$\psi(s) := \inf_{|x|=s} \frac{\int_{\mathbb{R}^d} \langle x, \nabla V(x-z) \rangle \nu_x(dz)}{|x|} > 0, \quad s \geq R_0, \quad (1.4)$$

*then $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha(r) = c\beta_\phi^{-1}(r)$ for some positive function ϕ satisfying*

$$\phi(x) = \frac{\int_{\mathbb{R}^d} \langle x, \nabla V(x-z) \rangle \nu_x(dz)}{(1+\sigma)|x|p_\sigma(|x|)}, \quad |x| \geq R_0, \quad (1.5)$$

with $\sigma > 0$ and

$$p_\sigma(r) := \frac{\int_{R_0}^r s^{1-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^s \psi(u) du \right] ds + 1}{r^{1-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^r \psi(u) du \right]}, \quad r \geq R_0. \quad (1.6)$$

(b) Suppose that $V \in C^2(\mathbb{R}^d)$, there exist $\delta \in (0, 1)$ and $R_0 > 0$ such that

$$\int_{\mathbb{R}^d} (\delta |\nabla V(x - z)|^2 - \Delta V(x - z)) \nu_x(dz) > 0, \quad |x| \geq R_0. \quad (1.7)$$

Then $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha(r) = c\beta_\phi^{-1}(r)$ for some positive function ϕ satisfying

$$\phi(x) = (1 - \delta) \int_{\mathbb{R}^d} (\delta |\nabla V(x - z)|^2 - \Delta V(x - z)) \nu_x(dz), \quad |x| \geq R_0.$$

Before moving on, let us give some comments on Theorem 1.1.

- (i) This result is obtained by using two different ways to find a suitable Lyapunov function. The first method is new and the second one is due to [12]. Since

$$\frac{\int_{\mathbb{R}^d} \langle x, \nabla V(x - z) \rangle \nu_x(dz)}{|x|} \geq \psi(|x|), \quad |x| \geq R_0,$$

we usually use

$$\frac{\psi(|x|)}{(1 + \sigma)p_\sigma(|x|)}, \quad |x| \geq R_0$$

instead of (1.5) as ϕ in specific calculation.

- (ii) To show the power of this result, in Section 3, we will apply it to the convolution with a discrete probability measure; see Example 3.1.
- (iii) In Remark 2.2, we will explain that the conditions in [12, Theorem 3.1 and Theorem 4.1] imply that presented above.

The function α , archived in Theorem 1.1 (a), depends on σ and ψ . To emphasize σ or both of them, we may use notations α_σ or $\alpha_{\psi, \sigma}$. In some cases, one has to choose σ carefully, see Examples 3.5 and 3.6. Moreover, we have the following proposition which concerns the rates of $\alpha(r)$ as $r \rightarrow 0$ provided by different σ and ψ .

Proposition 1.2. (a) Let ψ be a positive function on \mathbb{R}^+ with $\sigma_0 > 0$ and $M > 0$ such that

$$\inf_{t \geq M} \left[\frac{\sigma_0}{1 + \sigma_0} - \frac{\psi'(t)}{\psi^2(t)} + \frac{1 - d}{t\psi(t)} \right] > 0. \quad (1.8)$$

Then for $\sigma_2 > \sigma_1 \geq \sigma_0$, there exist $c_1 > 0$, $c_2 > 0$ and $r_0 > 0$ such that

$$c_1 \alpha_{\sigma_2}(r) \leq \alpha_{\sigma_1}(r) \leq c_2 \alpha_{\sigma_2}(r), \quad 0 < r \leq r_0,$$

where α_{σ_1} and α_{σ_2} are given by Theorem 1.1 (a) corresponding to σ_1 and σ_2 , respectively.

- (b) Assume that ψ_1 and ψ_2 are positive functions on \mathbb{R}^+ . If (1.8) holds for ψ_1 with some σ_0 , $M > 0$, and for fixed $\sigma > \sigma_0$, there exist $\epsilon_1 \in [0, \frac{1}{\sigma})$, $\epsilon_2 \in [0, \frac{\sigma - \sigma_0}{\sigma(1 + \sigma_0)}]$ and $R_1 > 0$ such that

$$(1 + \epsilon_1)\psi_1(r) \geq \psi_2(r) \geq (1 - \epsilon_2)\psi_1(r), \quad r \geq R_1,$$

then there exist positive constants c_{σ, ϵ_1} , c_{σ, ϵ_2} and r_0 such that

$$c_{\sigma, \epsilon_2} \alpha_{\psi_1, \sigma}(r) \leq \alpha_{\psi_2, \sigma}(r) \leq c_{\sigma, \epsilon_1} \alpha_{\psi_1, \sigma}(r), \quad r \leq r_0.$$

Indeed, a large class of functions satisfies (1.8). It is easy to check that (1.8) holds for $\psi(t) = \frac{c}{t^{1-p}}$ with $p > 0$, $c > 0$ when t is large enough or $\psi(t) = \frac{c}{t^{1-p} \log^q t}$ with $p > 0$, $q > 0$ when t is large enough, or $\psi(t) = \frac{c}{t}$ with $c > d - 2$ and t large enough. Combining Proposition 1.2 (a) and (b), if ψ_1 and ψ_2 are equivalent at infinity (i.e. $\lim_{t \rightarrow +\infty} \frac{\psi_2}{\psi_1} = 1$) with (1.8) holds for ψ_1 , then for all σ_1, σ_2 large than σ_0 , there exist $c_1, c_2 > 0$ and $r_0 > 0$ such that

$$c_1 \alpha_{\psi_2, \sigma_2}(r) \leq \alpha_{\psi_1, \sigma_1}(r) \leq c_2 \alpha_{\psi_2, \sigma_2}(r), \quad r \leq r_0,$$

i.e. ψ_1, ψ_2 provide two α with the same rate as r tending to 0.

We then apply the above-mentioned results to the convolution with compactly supported probability measures and obtain the following corollary.

Corollary 1.3. *Let ν be a probability measure on \mathbb{R}^d with compact support such that $R := \sup\{|z| : z \in \text{supp} \nu\} < \infty$.*

(a) *If there exists $R_0 > R$ such that*

$$\eta(s) := \inf_{|x|=s} \left(\langle \nabla V(x), x \rangle - R |\nabla V(x)| \right) > 0, \quad s \geq R_0, \quad (1.9)$$

*then $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha_\phi(r) = c\beta_\phi^{-1}(r)$ for some positive function ϕ satisfying*

$$\phi(x) = \frac{\psi(|x|)}{(\sigma + 1)p_\sigma(|x|)}, \quad |x| \geq R + R_0,$$

where $\psi(r) = \frac{1}{r} \inf_{r-R \leq s \leq r+R} \eta(s)$, $r \geq R_0$, $p_\sigma(r)$ is defined as in (1.6) and

$$\beta_\phi(r) = \mu\left(|x| \geq \varphi_\phi(r), |x| \geq R + R_0\right) \quad (1.10)$$

for r large enough.

(b) *If there is a constant $\delta \in (0, 1)$ and $R_0 > R$ such that*

$$\delta |\nabla V(x)|^2 - \Delta V(x) > 0, \quad |x| \geq R_0,$$

*then $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha(r) = c\beta_\phi^{-1}(r)$ for some positive function ϕ satisfying*

$$\phi(x) = \inf_{u \in B_R(x)} \left(\delta |\nabla V(u)|^2 - \Delta V(u) \right), \quad |x| \geq R + R_0$$

and $\beta_\phi(r)$ as in (1.10) for r large enough.

We also give some notes on this corollary.

- (i) As announced, we only need to estimate $\alpha(r)$ for r small enough, i.e. it suffices for us to explicit $\beta_\phi(r)$ for r large enough.
- (ii) When $\nu = \delta_0$, i.e. $R = 0$, Corollary 1.3 presents the criteria for the weak Poincaré inequality for μ . We will explain in Remark 3.7 that we can get refined results by using the Lyapunov function constructed here.

In [12], some sufficient conditions are presented for the stability under convolution with compactly supported probability measures. Here, we only need to investigate the change of the rate of $\alpha(r)$ for weak Poincaré inequalities as $r \rightarrow 0$, since the stability for the weak Poincaré inequality holds obviously. The corollary in the following is devoted to compare the function α obtained from probability measure μ and $\mu * \nu$ via Corollary 1.3

Corollary 1.4. *Assume that (1.9) holds for μ with $R = 0$ and some $R_0 > 0$. Let ψ_μ, η_μ (with subscript μ) be the corresponding functions defined in Corollary 1.3 for μ . Let ν be a probability measure such that part (a) in Corollary 1.3 holds for $\mu * \nu$ (we shall adapted the same notations used there such as ψ, η , etc.). If (1.8) holds for ψ_μ with some $M, \sigma_0 > 0$, and*

$$\eta_0 := \liminf_{r \rightarrow +\infty} \frac{\inf_{\{r-R \leq s \leq r+R\}} \eta(s)}{\eta_\mu(r)} > \frac{\sigma_0}{1 + \sigma_0},$$

then for $\sigma > \frac{\sigma_0}{\eta_0(1+\sigma_0)-\sigma_0}$, there exists $c_1 > 0, c_2 > 0$ and $r_0 > 0$ such that

$$c_1 \alpha_{\mu, \sigma}(r) \leq \alpha(r) \leq c_2 \alpha_{\mu, \sigma}(r), \quad r \leq r_0.$$

The rest parts of the paper are organized as follows. In the following section, we give the proof of Theorem 1.1 and Corollary 1.3 by using a Lyapunov condition. To show the power of this result, some explicit examples are studied in Section 3.

2 Proof of main results

Let L be a second order elliptic operator. To derive a weak Poincaré inequality, we need the following *general Lyapunov condition* (see [6]).

Hypothesis (L) There exist some continuous function $\phi : \mathbb{R}^d \rightarrow (0, +\infty)$ and $W \geq 1, b > 0, r_0 > 0$ such that

$$\frac{LW}{W} \leq -\phi + b1_{B_{r_0}}, \quad (2.1)$$

where $B_{r_0} := \{x \in \mathbb{R}^d : |x| \leq r_0\}$ is a ball with center 0 and radius r_0 .

Our first step is to prove that if Hypothesis (L) holds, then weak Poincaré inequality holds for some α . The following result is derived by combining [3, Theorem 4.6] and [3, Theorem 2.18]. For sake of completeness, we include it here.

Lemma 2.1. *Let $\mu(dx) = e^{-V(x)} dx$ be a probability measure on \mathbb{R}^d . Assume Lyapunov condition (L) holds for $L = \Delta - \nabla V$. Then the following weak Poincaré inequality*

$$\mu(f^2) \leq c_0 F^{-1}(r) \mu(|\nabla f|^2) + r \text{Osc}(f)^2, \quad r > 0$$

holds for some positive constant c_0 and $F(r) := \mu(\phi \leq \frac{1}{r})$.

Proof. For any $r > 0$ and $f \in C_b^1(\mathbb{R}^d)$ with $\mu(f) = 0$, we have

$$\begin{aligned}
\mu(f^2) &= \inf_{c \in \mathbb{R}} \mu(f - c)^2 \leq \int_{\{\phi > 1/r\}} (f - f(x_0))^2 d\mu + \int_{\{\phi \leq 1/r\}} (f - f(x_0))^2 d\mu \\
&\leq \int_{\{\phi > 1/r\}} (f - f(x_0))^2 d\mu + \mu(\{\phi \leq 1/r\}) \text{Osc}(f)^2 \\
&\leq r \int \phi (f - f(x_0))^2 d\mu + \mu(\{\phi \leq 1/r\}) \text{Osc}(f)^2 \\
&\leq -r \int \frac{LW}{W} (f - f(x_0))^2 d\mu + rb \int_{B_{r_0}} (f - f(x_0))^2 d\mu \\
&\quad + \mu(\{\phi \leq 1/r\}) \text{Osc}(f)^2,
\end{aligned} \tag{2.2}$$

where $x_0 \in \mathbb{R}^d$ will be specified later. Now we need to estimate the first two terms on the right hand side of the latter inequality: a global term and a local term. For the global term, by [4, Lemma 2.12], we have

$$- \int \frac{LW}{W} (f - f(x_0))^2 d\mu \leq \int |\nabla f|^2 d\mu. \tag{2.3}$$

For the local one, choose $x_0 \in \mathbb{R}^d$ such that $f(x_0) = \frac{1}{\mu(B_{r_0})} \left(\int_{B_{r_0}} f d\mu \right)$ and define $g = f - f(x_0)$. Then we have

$$\begin{aligned}
\int_{B_{r_0}} (f - f(x_0))^2 d\mu &= \int_{B_{r_0}} g^2 d\mu \leq \lambda_{r_0}^{-1} \int |\nabla g|^2 d\mu + \frac{1}{\mu(B_{r_0})} \left(\int_{B_{r_0}} g d\mu \right)^2 \\
&= \lambda_{r_0}^{-1} \mu(|\nabla g|^2) = \lambda_{r_0}^{-1} \mu(|\nabla f|^2),
\end{aligned} \tag{2.4}$$

where by [11, (4.3.5)],

$$\lambda_{r_0}^{-1} \leq \frac{4r_0^2}{\pi^2} \exp \left\{ \sup_{x, y \in B_{r_0}} (V(x) - V(y)) \right\} < \infty.$$

Now, taking (2.4) and (2.3) into (2.2) implies

$$\mu(f^2) \leq r(b\lambda_{r_0}^{-1} + 1) \int |\nabla f|^2 d\mu + \mu\left(\phi \leq \frac{1}{r}\right) \text{Osc}(f)^2. \tag{2.5}$$

Since ϕ is continuous and positive on \mathbb{R}^d and μ is a probability measure, we obtain

$$\lim_{r \rightarrow +\infty} \mu(\phi \leq 1/r) = 0.$$

Let

$$F(r) = \mu\left(\phi \leq \frac{1}{r}\right).$$

Then $F : (0, +\infty) \rightarrow (0, 1)$ is a continuous and decreasing function. Define $F^{-1}(s) = \inf\{r : F(r) \leq s\}$. Then

$$\alpha(r) := (b\lambda_{r_0}^{-1} + 1)F^{-1}(r)$$

is a function from $(0, +\infty)$ to $(0, +\infty)$ and the weak Poincaré inequality holds for such α . \square

By using this lemma, we complete the proof of the main result.

Proof of Theorem 1.1. Let $L_\nu = \Delta - \nabla V_\nu$. First, if the Lyapunov condition **(L)** holds for L_ν with some function ϕ , then

$$\begin{aligned} \mu * \nu \left(\phi \leq \frac{1}{r} \right) &\leq \mu * \nu(|x| \geq \varphi_\phi(r)) \leq \iint_{\{|x| \geq \varphi_\phi(r)\}} e^{-V(x-z)} \nu(dz) dx \\ &\leq \iint_{\{|x-z| \geq \frac{1}{2}\varphi_\phi(r)\}} e^{-V(x-z)} \nu(dz) dx + \iint_{\{|z| \geq \frac{1}{2}\varphi_\phi(r)\}} e^{-V(x-z)} \nu(dz) dx \\ &= \iint_{\{|x-z| \geq \frac{1}{2}\varphi_\phi(r)\}} e^{-V(x-z)} dx \nu(dz) + \int_{\{|z| \geq \frac{1}{2}\varphi_\phi(r)\}} \int e^{-V(x-z)} dx \nu(dz) \\ &= \mu \left(|x| \geq \frac{1}{2}\varphi_\phi(r) \right) + \nu \left(|z| \geq \frac{1}{2}\varphi_\phi(r) \right) \\ &= \beta_\phi(r). \end{aligned}$$

Hence, the weak Poincaré inequality holds for $\alpha(r) = c\beta_\phi^{-1}(r)$ with c a positive constant. We now turn to finding some suitable Lyapunov function ϕ .

In case (a), define the Lyapunov function as

$$W_\sigma(|x|) = \int_{R_0}^{|x|} \left(s^{1-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^s \psi(u) du \right] \right) ds + 1, \quad |x| \geq R_0.$$

Then for $|x| \geq R_0$,

$$\begin{aligned} \frac{(\Delta - \nabla V_\nu)W_\sigma(|x|)}{W_\sigma(|x|)} &= \frac{1}{W_\sigma(|x|)} (W'_\sigma(|x|)\Delta|x| + W''_\sigma(|x|)|\nabla|x||^2 - W'_\sigma(|x|) \langle \nabla V_\nu, \nabla|x| \rangle) \\ &= \frac{1}{W_\sigma(|x|)} \left\{ \frac{W'_\sigma(|x|)}{|x|} [(d-1) - \langle \nabla V_\nu, x \rangle] + W''_\sigma(|x|) \right\} \\ &= \frac{1}{W_\sigma(|x|)} \left\{ |x|^{-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^{|x|} \psi(u) du \right] [(d-1) - \langle \nabla V_\nu, x \rangle] \right. \\ &\quad \left. + |x|^{-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^{|x|} \psi(u) du \right] \left[(1-d) + \frac{\sigma}{\sigma+1} |x| \psi(|x|) \right] \right\} \\ &= \frac{|x|^{1-d}}{W_\sigma(|x|)} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^{|x|} \psi(u) du \right] \left[-\frac{\langle \nabla V_\nu, x \rangle}{|x|} + \frac{\sigma}{\sigma+1} \psi(|x|) \right]. \end{aligned}$$

By this and (1.4), we further obtain

$$\frac{(\Delta - \nabla V_\nu)W_\sigma(|x|)}{W_\sigma(|x|)} \leq -\frac{|x|^{1-d}}{(\sigma+1)W_\sigma(|x|)} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^{|x|} \psi(u) du \right] \frac{\langle \nabla V_\nu, x \rangle}{|x|}, \quad |x| > R_0.$$

Therefore, there exists a constant $b > 0$ such that

$$\frac{(\Delta - \nabla V_\nu)W_\sigma(|x|)}{W_\sigma(|x|)} \leq -\frac{\langle x, \nabla V_\nu(x) \rangle}{(1+\sigma)|x|p_\sigma(|x|)} 1_{\{|x| \geq R_0\}} + b 1_{\{|x| \leq R_0\}}.$$

In case (b), we consider a smooth function such that $W(x) = e^{(1-\delta)V_\nu(x)}$ for $|x| \geq R_0$ and $W(x) \geq 1$ for all $x \in \mathbb{R}^d$. Then,

$$\frac{(\Delta - \nabla V_\nu)W(x)}{W(x)} \leq -(1-\delta)(\delta|\nabla V_\nu|^2 - \Delta V_\nu) 1_{\{|x| \geq R_0\}} + b 1_{\{|x| \leq R_0\}}. \quad (2.6)$$

Now, it suffices to show that $\delta|\nabla V_\nu|^2 - \Delta V_\nu > 0$ when $|x| \leq R_0$. Indeed, by (1.7), for $|x| \geq R_0$,

$$\begin{aligned} \delta|\nabla V_\nu(x)|^2 - \Delta V_\nu(x) &= \int_{\mathbb{R}^d} (|\nabla V(x-z)|^2 - \Delta V(x-z))\nu_x(dz) - (1-\delta)|\nabla V_\nu(x)|^2 \\ &\geq \int_{\mathbb{R}^d} (\delta|\nabla V(x-z)|^2 - \Delta V(x-z))\nu_x(dz) \\ &> 0. \end{aligned}$$

Combining this with (2.6), we complete the proof of (b). \square

Remark 2.2. If $\psi(s) \geq c_0 > 0$ for $s \geq R_0$, then there exists $c_1 > 0$ such that

$$p_\sigma(r) = \frac{\int_{R_0}^r s^{1-d} \exp\left[\frac{c_0\sigma(s-R_0)}{\sigma+1}\right] ds + 1}{r^{1-d} \exp\left[\frac{c_0\sigma(r-R_0)}{\sigma+1}\right]} \leq c_1, \quad r \geq R_0.$$

Thus,

$$\frac{\psi(|x|)}{(1+\sigma)p_\sigma(|x|)} \geq c_0/c_1 \equiv \phi(|x|).$$

That means the conditions used in [12, Theorem 3.1] implies that used in this theorem.

Proof of Proposition 1.2. (i). It is no harm to assume that $R_0 > M$. Let

$$H(w, r) = \exp\left[w \int_{R_0}^r \psi(u) du\right], \quad \Psi(w, r) = \int_{R_0}^r s^{1-d} H(w, s) ds.$$

Fix $s > R_0$, we have

$$\frac{d}{dw} \left[\frac{\Psi(w, r) + 1}{r^{1-d} H(w, r)} \right] = \frac{-H(w, r) \left\{ \int_{R_0}^r s^{1-d} H(w, s) \left(\int_s^r \psi(u) du \right) ds + \int_{R_0}^r \psi(u) du \right\}}{r^{1-d} H^2(w, r)} < 0.$$

Let $w_1 = \frac{\sigma_1}{\sigma_1+1}$ and $w_2 = \frac{\sigma_2}{\sigma_2+1}$. Then

$$p_{\sigma_2}(r) = \frac{\Psi(w_2, r) + 1}{r^{1-d} H(w_2, r)} \leq \frac{\Psi(w_1, r) + 1}{r^{1-d} H(w_1, r)} = p_{\sigma_1}(r). \quad (2.7)$$

On the other hand, for r large enough, there exists $c_{w_1} > 0$,

$$\begin{aligned} \left[\frac{\Psi(w_1, r) + 1}{r^{1-d} H(w_1, r)} \right] \Big/ \left[\frac{\Psi(w_2, r) + 1}{r^{1-d} H(w_2, r)} \right] &= \frac{(\Psi(w_1, r) + 1) H(w_2 - w_1, r)}{\Psi(w_2, r) + 1} \\ &\leq \frac{c_{w_1} \Psi(w_1, r) H(w_2 - w_1, r)}{\Psi(w_2, r)}. \end{aligned}$$

We now turn to control the term on the right hand side of the latter inequality. Since

$$\begin{aligned} \frac{s^{1-d} H(w_2, s)}{H(w_2 - w_1, s) \psi(s)} &\geq \frac{s^{1-d}}{\psi(s)} \exp\left[w_1 \int_{R_0}^s \psi(u) du\right] - \frac{R_0^{1-d}}{\psi(R_0)} \\ &= \int_{R_0}^s t^{1-d} \left[w_1 - \frac{\psi'(t)}{\psi^2(t)} + \frac{1-d}{t\psi(t)} \right] \exp\left[w_1 \int_{R_0}^t \psi(u) du\right] dt \\ &\geq \theta \int_{R_0}^s t^{1-d} \exp\left[w_1 \int_{R_0}^t \psi(u) du\right] dt = \theta \Psi(w_1, s), \end{aligned}$$

where

$$\theta := \inf_{t \geq M} \left[w_1 - \frac{\psi'(t)}{\psi^2(t)} + \frac{1-d}{t\psi(t)} \right] \geq \inf_{t \geq M} \left[\frac{\sigma_0}{\sigma_0 + 1} - \frac{\psi'(t)}{\psi^2(t)} + \frac{1-d}{t\psi(t)} \right] > 0,$$

we obtain

$$\int_{R_0}^r \Psi(w_1, s) H(w_2 - w_1, s) \psi(s) ds \leq \frac{1}{\theta} \int_{R_0}^r s^{1-d} H(w_2, s) ds = \frac{\Psi(w_2, r)}{\theta}. \quad (2.8)$$

Moreover,

$$\begin{aligned} \Psi(w_2, r) &= \int_{R_0}^r s^{1-d} H(w_2 - w_1, s) H(w_1, s) ds \\ &\geq -(w_2 - w_1) \int_{R_0}^r \Psi(w_1, s) H(w_2 - w_1, s) \psi(s) ds \\ &\quad + H(w_2 - w_1, r) \Psi(w_1, r). \end{aligned}$$

By this and (2.8), we have

$$\frac{\theta + w_2 - w_1}{\theta} \Psi(w_2, r) \geq H(w_2 - w_1, r) \Psi(w_1, r).$$

Then

$$\frac{2\Psi(w_1, r) H(w_2 - w_1, r)}{\Psi(w_2, r)} \leq \frac{2(\theta + w_2 - w_1)}{\theta} < \frac{2(\theta + 1)}{\theta}.$$

Then, there exists a constant $c_{\sigma_1} \geq 1$ such that

$$p_{\sigma_2}(r) \leq p_{\sigma_1}(r) \leq c_{\sigma_1} p_{\sigma_2}(r), \quad \sigma_2 \geq \sigma_1, \quad r \geq R_0.$$

By the definition of α , we complete the proof of part (a).

(ii). Let $f, g \geq 0$. For $R_0 \geq R_1 \vee M$, define

$$\Phi_{f,g}(\delta) = \frac{\int_{R_0}^r \left(s^{1-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^s (g(u) + \delta f(u)) du \right] \right) ds + 1}{r^{1-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^r (g(u) + \delta f(u)) du \right]}.$$

Then

$$\frac{d\Phi_{f,g}}{d\delta} = - \frac{\frac{\sigma}{1+\sigma} \left\{ \int_{R_0}^r \left(s^{1-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^s (g(u) + \delta f(u)) du \right] \int_s^r f(u) du \right) ds + \int_{R_0}^r f(u) du \right\}}{r^{1-d} \exp \left[\frac{\sigma}{\sigma+1} \int_{R_0}^r (g(u) + \delta f(u)) du \right]} \leq 0.$$

So, for ψ_1, ψ_2 and ϵ_1, ϵ_2 given in Proposition 1.2 (b), we obtain

$$\Phi_{0, (1+\epsilon_1)\psi_1}(0) \leq \Phi_{0, \psi_2}(0) \leq \Phi_{0, (1-\epsilon_2)\psi_1}(0).$$

As it is easy to see from the conditions that

$$\frac{(1+\epsilon_1)\sigma}{1+\sigma} < 1, \quad \frac{(1-\epsilon_2)\sigma}{1+\sigma} \geq \frac{\sigma_0}{1+\sigma_0},$$

we can choose $\tilde{\sigma} > 0$ and $\hat{\sigma} > 0$ such that

$$\frac{(1+\epsilon_1)\sigma}{1+\sigma} = \frac{\tilde{\sigma}}{1+\tilde{\sigma}}, \quad \frac{(1-\epsilon_2)\sigma}{1+\sigma} = \frac{\hat{\sigma}}{1+\hat{\sigma}}.$$

Then

$$\frac{\hat{\sigma}}{\sigma} \phi_{\psi_1, \hat{\sigma}} \leq \phi_{\psi_2, \sigma} \leq \frac{\tilde{\sigma}}{\sigma} \phi_{\psi_1, \tilde{\sigma}}.$$

By the definition of α , we have

$$\frac{\sigma}{\hat{\sigma}} \alpha_{\psi_1, \hat{\sigma}}(r) \leq \alpha_{\psi_2, \sigma}(r) \leq \frac{\sigma}{\tilde{\sigma}} \alpha_{\psi_1, \tilde{\sigma}}(r).$$

Combining this with part (a), we complete the proof of part (b). \square

Proof of Corollary 1.3. Since ϕ is continuous and positive on \mathbb{R}^d , there exists $r_0 > 0$ such that

$$\{x : \phi(x) \geq 1/r_0\} \supseteq \{x : |x| \leq 2(R + R_0)\}.$$

Then

$$\varphi_\phi(r_0) \geq 2(R + R_0).$$

Since φ_ϕ is a non-decreasing function, we further obtain

$$\nu(|z| \geq 1/2\varphi_\phi(r)) \leq \nu(|z| \geq 1/2\varphi_\phi(r_0)) \leq \nu(|z| \geq R + R_0) = 0, \quad r \geq r_0.$$

On the other hand, for $r \geq r_0$,

$$\mu(|x| \geq 1/2\varphi_\phi(r)) = \mu(|x| \geq 1/2\varphi_\phi(r), |x| \geq R_0 + R).$$

Hence,

$$\beta_\phi(r) = \mu(|x| \geq 1/2\varphi_\phi(r), |x| \geq R_0 + R), \quad r \geq r_0$$

according to the definition of β_ϕ (see (1.3)).

In case (a). Let $|x| \geq R_0 + R$. It is clear that when $|z| \leq R$, we have

$$|x - z| \geq R_0. \tag{2.9}$$

It is easy to observe that

$$\begin{aligned} & \int_{\mathbb{R}^d} \langle x, \nabla V(x - z) \rangle \nu_x(dz) \\ &= \int_{\mathbb{R}^d} \left(\langle x - z, \nabla V(x - z) \rangle + \langle z, \nabla V(x - z) \rangle \right) \nu_x(dz) \\ &\geq \int_{\mathbb{R}^d} \left(\langle x - z, \nabla V(x - z) \rangle - R|\nabla V(x - z)| \right) \nu_x(dz) \\ &= \int_{\{|z| \leq R\}} \left(\langle x - z, \nabla V(x - z) \rangle - R|\nabla V(x - z)| \right) \nu_x(dz). \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{\int_{\mathbb{R}^d} \langle x, \nabla V(x - z) \rangle \nu_x(dz)}{|x|} \\ &\geq \frac{1}{|x|} \int_{\{|z| \leq R\}} (\langle x - z, \nabla V(x - z) \rangle - R|\nabla V(x - z)|) \nu_x(dz) \\ &\geq \frac{1}{|x|} \inf_{|x| - R \leq s \leq |x| + R} \eta(s) = \psi(|x|). \end{aligned}$$

Choosing $\phi : \mathbb{R}^d \rightarrow (0, \infty)$ such that for $|x| \geq R + R_0$,

$$\phi(x) = \frac{\psi(|x|)}{(\sigma + 1)p_\sigma(|x|)},$$

we then complete the proof of (a) due to Theorem 1.1.

In case (b). For $|x| \geq R_0 + R$, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\delta |\nabla V|^2(x - z) - \Delta V(x - z) \right) \nu_x(dz) \\ &= \int_{\{|z| \leq R\}} \left(\delta |\nabla V|^2(x - z) - \Delta V(x - z) \right) \nu_x(dz) \\ &\geq \inf_{u \in B_R(x)} \left(\delta |\nabla V|^2(u) - \Delta V(u) \right) > 0, \end{aligned}$$

which leads to complete the proof. \square

Proof of Corollary 1.4 Since $\psi_\mu(r) = \eta_\mu(r)/r$, $\psi(r) = \frac{1}{r} \inf_{\{r-R \leq s \leq r+R\}} \eta(s)$, and $\eta_\mu(r) \geq \eta(r)$, we obtain that

$$1 \geq \liminf_{r \rightarrow +\infty} \frac{\psi(r)}{\psi_\mu(r)} = \eta_0 > \frac{\sigma_0}{1 + \sigma_0}.$$

For $\sigma > \frac{\sigma_0}{\eta_0(1+\sigma_0)-\sigma_0} (\geq \sigma_0)$, it is clear that $\eta_0 > \frac{\sigma_0(1+\sigma)}{\sigma(1+\sigma_0)}$, i.e. $\frac{\sigma_0(1+\sigma)}{\sigma(1+\sigma_0)\eta_0} < 1$. So there exist $R_1 > 0$ such that

$$\psi_\mu(r) \geq \psi(r) \geq \eta_0 \frac{\sigma_0(1+\sigma)}{\sigma(1+\sigma_0)\eta_0} \psi_\mu(r) = \left(1 - \frac{\sigma - \sigma_0}{\sigma(1+\sigma_0)} \right) \psi_\mu(r), \quad r \geq R_1.$$

Then the corollary is a direct consequence of Proposition 1.2. \square

3 Examples

To see the power of Theorem 1.1, we present below an example where the support of ν is unbounded and disconnected. *From now on, the constants C and c in the conclusions and their proofs may change in different lines.*

Example 3.1. Let $d = 1$, $V(x) = c + (1 + x^2)^{\frac{\delta}{2}}$, $0 < \delta < 1$ and

$$\nu(dz) = \frac{1}{\gamma} \sum_{i \in \mathbb{Z}} \frac{\delta_i(dz)}{1 + |z|^{1+p}}, \quad p > 0,$$

where

$$c = \log \int_{\mathbb{R}} e^{-(1+x^2)^{\frac{\delta}{2}}} dx, \quad \gamma = \sum_{i \in \mathbb{Z}} \frac{1}{1 + |i|^{1+p}}.$$

Then the weak Poincaré inequality for $\mu * \nu$ holds with $\alpha(s) = Cs^{-2/p}$ for some $C > 0$.

To prove this result, we need the following lemma.

Lemma 3.2. Let $d = 1$, $V(x) = c + (1 + x^2)^{\frac{\delta}{2}}$, $0 < \delta < 1$ and

$$\tilde{\nu}(\mathrm{d}z) = \frac{1}{\gamma} \frac{1}{1 + |z|^{1+p}} \mathrm{d}z,$$

where

$$c = \log \int_{\mathbb{R}} e^{-(1+x^2)^{\frac{\delta}{2}}} \mathrm{d}x, \quad \gamma := \int_{\mathbb{R}} \frac{1}{1 + |z|^{1+p}} \mathrm{d}z, \quad p > 0.$$

Then the weak Poincaré inequality for $\mu * \tilde{\nu}$ holds with $\tilde{\alpha}(s) = Cs^{-2/p}$ for some $C > 0$.

Proof. For $x > 0$,

$$\begin{aligned} \frac{\int_{\mathbb{R}} x V'(x-z) \tilde{\nu}_x(\mathrm{d}z)}{|x|} &= \int_{\mathbb{R}} V'(x-z) \tilde{\nu}_x(\mathrm{d}z) = \int_{\mathbb{R}} \frac{(x-z)}{[1 + (x-z)^2]^{1-\frac{\delta}{2}}} \tilde{\nu}_x(\mathrm{d}z) \\ &= \frac{\int_{\mathbb{R}} (x-z) e^{-(1+(x-z)^2)^{\frac{\delta}{2}}} [1 + (x-z)^2]^{\frac{\delta}{2}-1} (1 + |z|^{1+p})^{-1} \mathrm{d}z}{\int_{\mathbb{R}} e^{-(1+(x-z)^2)^{\frac{\delta}{2}}} (1 + |z|^{1+p})^{-1} \mathrm{d}z}. \end{aligned}$$

It is clear that

$$\begin{aligned} &\int_{\mathbb{R}} \frac{(x-z) e^{-(1+(x-z)^2)^{\frac{\delta}{2}}}}{[1 + (x-z)^2]^{1-\frac{\delta}{2}} (1 + |z|^{1+p})} \mathrm{d}z \\ &= \int_0^\infty \frac{u e^{-(1+u^2)^{\frac{\delta}{2}}}}{(1+u^2)^{1-\delta/2}} \left[\frac{1}{1 + |x-u|^{p+1}} - \frac{1}{1 + |x+u|^{p+1}} \right] \mathrm{d}u \\ &= \int_0^\infty \frac{u e^{-(1+u^2)^{\frac{\delta}{2}}}}{(1+u^2)^{1-\delta/2}} \left[\frac{|x+u|^{1+p} - |x-u|^{1+p}}{(1 + |x-u|^{p+1})(1 + |x+u|^{p+1})} \right] \mathrm{d}u. \end{aligned} \quad (3.1)$$

Since

$$2(p+1)u|x-u|^p 1_{\{x>u\}} \leq |x+u|^{p+1} - |x-u|^{p+1} \leq 2(p+1)u|x+u|^p,$$

we obtain

$$\begin{aligned} \frac{x^{p+2} \left(|x+u|^{1+p} - |x-u|^{1+p} \right)}{(1 + |x-u|^{p+1})(1 + |x+u|^{p+1})} &\leq \frac{2u(p+1)x^{p+2}|x+u|^p}{(1 + |x-u|^{p+1})(1 + |x+u|^{p+1})} \\ &\leq \frac{2^{p+1}(p+1) \left(|x-u|^{p+1} + u^{p+1} \right) u|x+u|^{p+1}}{(1 + |x-u|^{p+1})(1 + |x+u|^{p+1})} \\ &\leq 2^{p+1}(p+1) \left(u + u^{p+2} \right) \end{aligned}$$

and

$$\frac{x^{p+2} \left(|x+u|^{1+p} - |x-u|^{1+p} \right)}{(1 + |x-u|^{p+1})(1 + |x+u|^{p+1})} \geq \frac{2(p+1)u x^{p+2} |x-u|^p 1_{\{x>u\}}}{(1 + |x-u|^{p+1})(1 + |x+u|^{p+1})}.$$

Combining these with (3.1) and using the dominated convergence theorem, we get

$$\begin{aligned} &\lim_{x \rightarrow +\infty} \int_{\mathbb{R}} \frac{x^{p+2} (x-z) e^{-[1+(x-z)^2]^{\frac{\delta}{2}}}}{[1 + (x-z)^2]^{1-\frac{\delta}{2}} (1 + |z|^{1+p})} \mathrm{d}z \\ &\leq 2(p+1) \int_0^\infty \frac{u^2 e^{-(1+u^2)^{\frac{\delta}{2}}}}{(1+u^2)^{1-\delta/2}} \mathrm{d}u < \infty, \end{aligned}$$

and

$$\begin{aligned}
& \lim_{x \rightarrow +\infty} \int_{\mathbb{R}} \frac{x^{p+2}(x-z) e^{-(1+(x-z)^2)^{\frac{\delta}{2}}}}{[1+(x-z)^2]^{1-\frac{\delta}{2}}(1+|z|^{1+p})} dz \\
& \geq 2(p+1) \lim_{x \rightarrow +\infty} \int_0^x \frac{u^2 e^{-(1+u^2)^{\frac{\delta}{2}}} x^{p+2}|x-u|^p}{(1+|x-u|^{p+1})(1+|x+u|^{p+1})(1+u^2)^{1-\delta/2}} du \\
& = 2(p+1) \int_0^\infty \frac{u^2 e^{-(1+u^2)^{\frac{\delta}{2}}}}{(1+u^2)^{1-\delta/2}} du.
\end{aligned}$$

Similarly, we have

$$\lim_{x \rightarrow +\infty} \int_{\mathbb{R}} \frac{x^{p+1} e^{-[1+(x-z)^2]^{\frac{\delta}{2}}}}{1+|z|^{1+p}} dz = \lim_{x \rightarrow +\infty} \int_{\mathbb{R}} \frac{x^{p+1} e^{-(1+u^2)^{\delta/2}}}{1+|x-u|^{p+1}} du = \int_{\mathbb{R}} e^{-(1+u^2)^{\delta/2}} du,$$

due to the following inequality and the dominated convergence theorem,

$$\frac{x^{p+1}}{1+|x-u|^{p+1}} \leq \frac{2^p(|x-u|^{p+1} + |u|^{p+1})}{1+|x-u|^{p+1}} \leq 2^p(1+|u|^{p+1}).$$

Then, we conclude that there exist $c > 0$, $R > 0$ such that

$$\frac{\int_{\mathbb{R}} x V'(x-z) \tilde{\nu}_x(dz)}{|x|} \geq \frac{c}{|x|}, \quad |x| \geq R.$$

Then

$$p_\sigma(r) = \frac{\int_{R_0}^r e^{c \frac{\sigma}{1+\sigma} \log u} du + 1}{e^{c \frac{\sigma}{1+\sigma} \log r}} \geq Cr$$

for some $C > 0$. Let $\phi(x) = \frac{C}{|x|^2}$. Then

$$\beta_\phi(r) = Cr^{p/2}$$

for some $C > 0$. \square

Proof of Example 3.1. To get the weak Poincaré inequality for $\mu * \nu$, we need to compare it with $\mu * \tilde{\nu}$ defined in Lemma 3.2. Let $k \in \mathbb{Z}$. For $z \in [k-1/2, k+1/2]$, we have

$$\begin{aligned}
1+(x-z)^2 &= 1+(x-k)^2 + 2(x-k)(k-z) + (k-z)^2 \\
&\leq 1+(x-k)^2 + |x-k| + (k-z)^2 \\
&\leq 3/2 [1+(x-k)^2] + 1/4
\end{aligned}$$

and

$$\begin{aligned}
1+(x-z)^2 &= 1+(x-k)^2 + 2(x-k)(k-z) + (k-z)^2 \\
&\geq 1+(x-k)^2 - |x-k| + (k-z)^2 \\
&\geq 1/2 [1+(x-k)^2].
\end{aligned}$$

So,

$$1/2 \leq \frac{1+(x-z)^2}{1+(k-z)^2} \leq 3/2 + 1/4 = 7/4. \quad (3.2)$$

On the other hand, there exist positive constants $c_{p,1}$, $c_{p,2}$ such that

$$\frac{c_{p,1}}{1 + |k|^{1+p}} \leq \frac{1}{1 + |z|^{1+p}} \leq \frac{c_{p,2}}{1 + |k|^{1+p}}, \quad z \in [k - 1/2, k + 1/2], \quad k \in \mathbb{Z}. \quad (3.3)$$

It follows from (3.2) and (3.3) that there exist $c_1, c_2 > 0$ such that

$$\begin{aligned} c_1 \sum_{k \in \mathbb{Z}} \frac{e^{-(1+(x-k)^2)^{\delta/2}}}{1 + |k|^{1+p}} &\leq \int_{\mathbb{R}} \frac{e^{-[1+(x-z)^2]^{\delta/2}}}{1 + |z|^{1+p}} dz = \sum_{k \in \mathbb{Z}} \int_{k-1/2}^{k+1/2} \frac{e^{-[1+(x-z)^2]^{\delta/2}}}{1 + |z|^{1+p}} dz \\ &\leq c_2 \sum_{k \in \mathbb{Z}} \frac{e^{-[1+(x-k)^2]^{\delta/2}}}{1 + |k|^{1+p}}. \end{aligned}$$

Thus, according to the definition of convolution measures (see (1.2)), there exist positive constants \tilde{c}_1, \tilde{c}_2 such that

$$\tilde{c}_1 \mu * \tilde{\nu} \leq \mu * \nu \leq \tilde{c}_2 \mu * \tilde{\nu}.$$

If the weak Poincaré inequality holds for $\mu * \tilde{\nu}$ with some function $\tilde{\alpha}$, then

$$\mu * \nu(f^2) \leq \tilde{c}_2 \mu * \tilde{\nu}(f^2) \leq \frac{\tilde{c}_2}{\tilde{c}_1} \tilde{\alpha}(r) \mu * \nu(|\nabla f|^2) + \tilde{c}_2 r \text{Osc}^2(f).$$

Therefore, the weak Poincaré inequality holds for $\mu * \nu$ with $\alpha(r) = \frac{\tilde{c}_2}{\tilde{c}_1} \tilde{\alpha}(\frac{r}{\tilde{c}_2})$. It is clear that α and $\tilde{\alpha}$ have the same order when $r \rightarrow 0$. So by Lemma 3.2, we get that the weak Poincaré inequality holds for $\mu * \nu$ with $\alpha(s) = Cs^{-2/p}$ for some $C > 0$. \square

Then, we present the following examples to illustrate Corollary 1.3.

Example 3.3. Let $V(x) = c + |x|^p$ for some $0 < p < 1$, $\mu(dx) = e^{-V(x)} dx$. Then for any compact probability measure ν and $R := \{|z| : z \in \text{supp } \nu\}$, the weak poincaré inequality for $\mu * \nu$ holds with

$$\alpha(s) = C \left[1 + \log \left(1 + \frac{1}{s} \right) \right]^{\frac{2(1-p)}{p}}$$

for some constant $C > 0$.

Proof. a) *Method 1.* Let $v(s) = c + s^p$. Then

$$\eta(s) = \inf_{|x|=s} (\langle \nabla V(x), x \rangle - R |\nabla V(x)|) = v'(s)(s - R).$$

It is easy to obtain that there exists $R_0 > 0$ such that for $|x| \geq R_0$,

$$\inf_{|x|-R \leq s \leq |x|+R} v'(s)(s - R) \leq c|x|^p.$$

Therefore, $\psi(|x|) := c|x|^{p-1}$. Then

$$p_\sigma(r) = \frac{\int_{R_0}^r u^{1-d} e^{c \frac{\sigma}{1+\sigma} u^p} du + 1}{r^{1-d} e^{c \frac{\sigma}{1+\sigma} r^p}} \leq C \frac{r^{2-p-d} e^{c \frac{\sigma}{1+\sigma} r^p}}{r^{1-d} e^{c \frac{\sigma}{1+\sigma} r^p}} = Cr^{1-p}.$$

Let

$$\phi(|x|) = Cp|x|^{2(p-1)}, \quad |x| \geq R_0.$$

Then for $r \geq r_0$ such that $\varphi_\phi(r_0) \geq 2R_0$,

$$\beta_\phi(r) = \mu\left(2|x| \geq \varphi_\phi(r), |x| \geq R_0\right) = C \int_{cr^{\frac{1}{2(2-p)}} \vee R_0}^{\infty} e^{-u^p} u^{d-1} du \leq C e^{-cr^{\frac{p}{2(1-p)}}} r^{\frac{d-p}{2(1-p)}}.$$

Therefore, there exists constant $C > 0$ such that

$$\alpha(s) = C \left[1 + \log \left(1 + \frac{1}{s} \right) \right]^{\frac{2(1-p)}{p}}.$$

b) *Method 2.* For $\delta > 0$ and $|x| > 0$,

$$\begin{aligned} \delta |\nabla V(x)|^2 - \Delta V(x) &= \delta |\nabla |x|^p|^2 - \Delta |x|^\delta \\ &= \delta p^2 |x|^{2(p-1)} - p(d+p-2) |x|^{p-2}. \end{aligned}$$

So, there exists constant $R_0 > 0$ such that

$$\delta |\nabla V(x)|^2 - \Delta V(x) \leq c |x|^{2(p-1)}, \quad |x| \geq R_0.$$

Then, for $|x| \geq R + R_0$,

$$\inf_{|x|-R \leq |u| \leq |x|+R} |u|^{2(p-1)} \leq c |x|^{2(p-1)},$$

which implies that for $|x| \geq R + R_0$,

$$\phi(x) = c |x|^{2(p-1)}.$$

The rest part of the proof is similar, we omit it. \square

Remark 3.4. Though when $p < 1$, the function $V(x) = c + |x|^p$ is not in $C^1(\mathbb{R}^d)$ at point 0 (it is the same case for examples in the following), we will explain that by a bounded perturbation, we can simply assume $V \in C^2(\mathbb{R}^d)$ as in the proof of Example 3.3. Take $l > 0$ and $\tilde{V} \in C^2(\mathbb{R}^d)$ such that

$$\text{supp}\{\tilde{V} - V\} \subset B_l(0).$$

Let $\tilde{\mu} = \tilde{C} e^{-\tilde{V}(x)} dx$ with a constant $\tilde{C} > 0$ such that $\tilde{\mu}$ is a probability measure. Then there exists $c > 0$ such that

$$c\tilde{\mu} * \nu \leq \mu * \nu \leq \frac{1}{c}\tilde{\mu} * \nu.$$

If the weak Poincaré inequality holds for $\tilde{\mu} * \nu$ with some function $\tilde{\alpha}$, then

$$\mu * \nu(f^2) \leq \frac{1}{c}\tilde{\mu} * \nu(f^2) \leq \frac{1}{c^2}\tilde{\alpha}(r)\mu * \nu(|\nabla f|^2) + \frac{r}{c}\text{Osc}^2(f).$$

Therefore, the weak Poincaré inequality holds for $\mu * \nu$ with $\alpha(r) = \frac{1}{c^2}\tilde{\alpha}(cr)$. It is clear that α and $\tilde{\alpha}$ have the same order when $r \rightarrow 0$.

Example 3.5. For $p > 0$, let $V(x) = c + (d+p) \log(1 + |x|)$. Then for any compact probability measure ν and $R := \{|z| : z \in \text{supp } \nu\}$, the weak poincaré inequality for $\mu * \nu$ holds with

$$\alpha(s) = Cs^{-\frac{2}{p}}$$

for some constant $C > 0$.

Proof. a) *Method 1.* Let $v(s) = c + (d + p) \log(1 + s)$. It is easy to obtain that

$$\inf_{|x|-R \leq s \leq |x|+R} v'(s)(s - R) = (d + p) \left(1 - \frac{R + 1}{|x| - R} \right).$$

Choose $\sigma > 0$ such that

$$\frac{\sigma(p + d)}{\sigma + 1} > d - 1.$$

Then there exists $R_0 > 0$ such that for $|x| \geq R_0$, $1 - \frac{R+1}{|x|-R} \geq \delta > 0$ with

$$\frac{\sigma\delta(p + d)}{\sigma + 1} > d - 1. \quad (3.4)$$

Set $\psi(|x|) = \delta(p + d)|x|^{-1}$. Then we have

$$\begin{aligned} p_\sigma(r) &= \frac{\int_{R_0}^r s^{1-d} \exp \left[\frac{\sigma\delta(p+d)}{\sigma+1} \int_{R_0}^s \frac{1}{1+u} du \right] ds + 1}{r^{1-d} \exp \left[\frac{\sigma\delta(p+d)}{\sigma+1} \int_{R_0}^r \frac{1}{1+u} du \right]} \\ &\leq \frac{C \int_{R_0}^r s^{1-d} (1 + s)^{\frac{\sigma\delta(p+d)}{\sigma+1}} ds + 1}{r^{1-d} \exp \left[\frac{\sigma\delta(p+d)}{\sigma+1} (\ln(1 + r) - \ln(1 + R_0)) \right]}. \end{aligned} \quad (3.5)$$

According to (3.4), we have

$$\begin{aligned} \int_{R_0}^r s^{1-d} (1 + s)^{\frac{\sigma\delta(p+d)}{\sigma+1}} ds + 1 &\leq C(r + 1)^{\frac{\sigma\delta(p+d)}{\sigma+1} + 2 - d}, \\ r^{1-d} \exp \left[\frac{\sigma\delta(p+d)}{\sigma+1} (\ln(1 + r) - \ln(1 + R_0)) \right] &\geq C(r + 1)^{\frac{\sigma\delta(p+d)}{\sigma+1} + 1 - d}. \end{aligned}$$

It follows from (3.5) that $p_\sigma(r) \leq C(r + 1)$ for some $C > 0$. Let

$$\phi(|x|) = \frac{C}{(|x| + 1)^2}, \quad |x| \geq R_0.$$

Then for $r \geq r_0$ such that $\varphi_\phi(r_0) \geq 2R_0$,

$$\begin{aligned} \beta_\phi(r) &= \mu(2|x| \geq \varphi_\phi(r), |x| \geq R_0) \leq \mu(|x| \geq ((Cr)^{1/2} - 1) \vee R_0) \\ &\leq C \int_{[(Cr)^{1/2} - 1] \vee R_0}^{\infty} \frac{1}{(1 + s)^{p+1}} ds \\ &\leq Cr^{-\frac{p}{2}}. \end{aligned}$$

Therefore,

$$\alpha(s) = Cs^{-\frac{2}{p}}$$

for some constant $C > 0$.

b) *Method 2.* There exists $\delta \in (0, 1)$ such that

$$\begin{aligned} \delta|\nabla V(x)|^2 - \Delta V(x) &= \delta|(d + p)\nabla \log(1 + |x|)|^2 + (d + p)\Delta \log(1 + |x|) \\ &= \frac{\delta(d + p)^2}{(1 + |x|)^2} - \frac{(d + p)(d - 1)}{(1 + |x|)|x|} + \frac{d + p}{(1 + |x|)^2} \end{aligned}$$

$$\leq \frac{c}{(1+|x|)^2}$$

for some positive constant $c > 0$. Therefore, for $|x| > R$,

$$\phi(|x|) = \inf_{|x|-R \leq |u| \leq |x|+R} \delta |\nabla V(u)|^2 - \Delta V(u) \leq \frac{c}{(1+|x|)^2}.$$

Then the rest step is similar to a). \square

Example 3.6. Let $p > 1$ and $V(x) = c + d \log(1+|x|) + p \log \log(e+|x|)$. Then for any compact probability measure ν and $R := \{|z| : z \in \text{supp } \nu\}$, the weak poincaré inequality for $\mu * \nu$ holds with

$$\alpha(r) = c_1 \exp[c_2 r^{-1/(p-1)}]$$

for some constant $c_1, c_2 > 0$.

Proof. a) *Method 1.* Let $v(s) = c + d \log(1+s) + p \log \log(e+s)$. Then

$$\inf_{|x|-R \leq s \leq |x|+R} v'(s)(s-R) \geq \frac{d(|x|-2R)}{1+|x|+R} + \frac{p}{\log(e+|x|+R)} \frac{|x|-2R}{e+|x|+R}.$$

Choose $\sigma > 0$ and $\delta \in (0, 1)$ such that

$$\frac{\sigma \delta d}{\sigma + 1} > d - 1.$$

Then there exists $R_0 > 0$ such that for $|x| \geq R_0$,

$$\frac{d(|x|-2R)}{1+|x|+R} + \frac{p}{\log(e+|x|+R)} \frac{|x|-2R}{e+|x|+R} \geq d\delta.$$

As calculating in the proof of Example 3.5, there exists some constant $c > 0$ such that

$$p_\sigma(|x|) = c|x|, \quad |x| \geq R_0.$$

Define $\phi(|x|) = \frac{c}{(1+|x|)^2}$, $|x| \geq R_0$. Then for $r \geq r_0$ such that $\varphi_\phi(r_0) \geq 2R_0$,

$$\begin{aligned} \beta_\phi(r) &= \mu(2|x| \leq \varphi_\phi(r)) \leq \mu\left(c \log(e+|x|) \geq \log(r+1)\right) \\ &\leq c(\log(1+r))^{-(p-1)}, \end{aligned}$$

which leads to complete the proof directly.

b) *Method 2.* There exists $\delta \in (0, 1)$ such that

$$\begin{aligned} \delta |\nabla V(x)|^2 - \Delta V(x) &= \delta |d \nabla \log(1+|x|) + p \nabla \log \log(e+|x|)|^2 \\ &\quad - d \Delta \log(1+|x|) - p \Delta \log \log(e+|x|) \\ &= \left| \frac{d(d-1)}{(1+|x|)} + \frac{p}{(e+|x|) \log(e+|x|)} \right|^2 \\ &\quad - \frac{d(d-1)}{(1+|x|)|x|} + \frac{d}{(1+|x|)^2} - \frac{p(d-1)}{|x|(e+|x|) \log(e+|x|)} \end{aligned}$$

$$\begin{aligned}
& + \frac{p(\log(e+|x|) + 1)}{(e+|x|)^2(\log(e+|x|))^2} \\
& \leq \frac{c}{(1+|x|)^2}
\end{aligned}$$

for some positive constant c . Therefore, for $|x| > R$,

$$\phi(|x|) = \inf_{|x|-R \leq |u| \leq |x|+R} \delta |\nabla V(u)|^2 - \Delta V(u) \leq \frac{c}{(1+|x|)^2}.$$

Then the rest step is similar to a). \square

Remark 3.7. When $\nu = \delta_0$, i.e. $R = 0$. Examples 3.3–3.6 have been treated in [11]. Compared with [11], these results, presented above, are more precise. It would like to indicate that in Example 3.5, this weak Poincaré inequality implies the exact main order of $\mu * \nu(\rho > N)$ as $N \rightarrow \infty$ by [11, Corollary 4.2.2 (1)].

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